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Topological characterization of nonsingular real algebraic G-surfaces

Sung Sook Kim [*,a], Mikiya Masuda [b]

[a] Department of Applied Mathematics, Paichai University, Taejon, 302-735, South Korea
[b] Department of Mathematics, Osaka City University, Osaka 558, Japan

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Abstract

In this paper we prove that a smooth G-manifold of dimension 2 is equivariantly diffeomorphic to an affine G-variety if and only if it is equivariantly diffeomorphic to the interior of a compact smooth G-manifold.

Key words: Real algebraic variety; Interior manifold conjecture; Nash–Tognoli Theorem; Equivariant diffeomorphism

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0. Introduction

Let G be a compact Lie group and Ω be a real G-module. A *real (affine) algebraic G-variety* is a G-invariant set

$$X = \{x \in \Omega \mid p_1(x) = \cdots = p_m(x) = 0\}$$

for polynomials $p_1, \ldots, p_m : \Omega \to \mathbb{R}$. If a real algebraic G-variety is nonsingular, then it is a smooth G-manifold without boundary. We say that a smooth G-manifold is *algebraically realized* if it is equivariantly diffeomorphic to a nonsingular real algebraic G-variety.

We note that not every smooth G-manifold without boundary is algebraically realized. In fact, if a smooth G-manifold is algebraically realized, then it must be *compactified*, i.e., equivariantly diffeomorphic to the interior $W - \partial W$ of a compact smooth G-manifold W [13, 14].

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Interior Manifold Conjecture (cf. [10]). *A smooth G-manifold is algebraically realized if and only if it is compactified.*

In the nonequivariant situation this has been proved by Akbulut and King [1]. It is not difficult to prove the conjecture in dimension 1. In this paper we prove it in dimension 2, i.e., we have

Theorem 0.1. *A smooth G-manifold of dimension* 2 *is algebraically realized if and only if it is compactified.*

A closed smooth G-manifold is obviously compactified. The conjecture above restricted to closed smooth G-manifolds is of particular interest and known as the Nash–Tognoli Theorem [12,17] when G is trivial. In case G is nontrivial, it is not solved completely but there are some positive solutions with some restrictions on the acting group G or on the action [5–9].

Dovermann, Knop and Suh [5] have already proved Theorem 0.1 for closed smooth orientable surfaces with orientation preserving actions of finite groups. They used complex algebraic geometry but our method is purely topological and based on

Theorem 0.2 [8, Theorem 1.3]. *If a closed smooth G-manifold is G-cobordant to a nonsingular real algebraic G-variety, then it is algebraically realized.*

Here is an outline of the proof of Theorem 0.1 in the case where G is finite and the smooth G-manifold is closed. Let M be a closed smooth G-manifold of dimension 2. By Theorem 0.2 it suffices to find a nonsingular real algebraic G-variety which is G-cobordant to M. A result of Stong [16] together with some observation reduces the problem to the case where G is a 2-group. If G is a 2-group, then it has a central element τ of order 2. Using τ, one can find a G-cobordism between M and $P(\nu(M^\tau) \oplus \underline{\mathbb{R}})$ the total space of the real projective bundle of the G-vector bundle $\nu(M^\tau) \oplus \underline{\mathbb{R}}$ where $\nu(M^\tau)$ denotes the G-normal bundle of the τ-fixed point set M^τ and $\underline{\mathbb{R}}$ the trivial G-vector bundle over M^τ with fiber $\mathbb{R}$. Since dim $M = 2$, M^τ is a disjoint union of circles and isolated points. We classify equivariant line bundles over a circle and see that $P(\nu(M^\tau) \oplus \underline{\mathbb{R}})$ is algebraically realized. Then it follows from Theorem 0.2 that M is algebraically realized.

Throughout this paper G will denote a compact Lie group unless otherwise stated.

1. Equivariant line bundles over S^1

Let S^1 denote the unit circle in $\mathbb{R}^2$ with the standard metric. We classify equivariant line bundles over S^1. In the nonequivariant case there are, up to isomorphism, only two line bundles over S^1, the trivial bundle and the Hopf bundle. In the equivariant case we obtain a similar result.

Proposition 1.1. *A smooth G-line bundle $L \to S^1$ is equivariantly isomorphic to a product bundle $S(V) \times \delta \to S(V)$ or $S(V) \times_{\mathbb{Z}_2} \delta \to S(V)/\mathbb{Z}_2 = P(V)$ according as the G-line bundle $L \to S^1$ is trivial or not when we forget the action. Here $S(V)$ denotes the unit circle of a real 2-dimensional orthogonal G-module V, δ a real 1-dimensional G-module and $\mathbb{Z}_2$ acts on $S(V)$ and δ as scalar multiplication.*

Proof. Since G is compact, L admits a G-invariant fiber metric. We choose one and fix it.

Case 1: *Suppose $L \to S^1$ is trivial when we forget the action*. Then we can identify L with $S^1 \times \mathbb{R}$ and may assume that the G-action on $S^1 \times \mathbb{R}$ preserves the standard fiber metric on $S^1 \times \mathbb{R}$. Moreover we may assume that the G-action on S^1 is linear [15, Theorem 2.0]. Hence one can express the action of $g \in G$ on $S^1 \times \mathbb{R}$ as

$$(x, v) \to \big(\rho(g)(x), \phi_g(x)v\big) \quad \text{for } (x, v) \in S^1 \times \mathbb{R}$$

where $\rho : G \to O(2)$ is a homomorphism, $\rho(g)$ acts on S^1 in the standard way and $\phi_g(x)$ is a scalar. Since the action of g preserves the standard metric on $S^1 \times \mathbb{R}$, $\phi_g(x)$ must be ± 1. The map $\phi_g : S^1 \to \{\pm 1\} = \mathbb{Z}_2$ is continuous and S^1 is connected, so $\phi_g(x)$ is independent of $x \in S^1$. Hence we have a homomorphism $\phi : G \to \mathbb{Z}_2$ given by a $g \to \phi_g$. This proves the statement of the proposition in the case where $L \to S^1$ is trivial when we forget the action (in fact, V and δ are G-modules determined by ρ and ϕ respectively).

Case 2: *Suppose $L \to S^1$ is nontrivial when we forget the action*. Then the total space of its sphere bundle, denoted $S(L)$, is diffeomorphic to S^1 and the projection $\pi : S(L) \to S^1$ is an equivariant double covering map. Since the induced G-line bundle by π from $L \to S^1$ is trivial when we forget the action, we apply the result of Case 1 to the induced G-line bundle to get the statement of the proposition. $\square$

2. Projective bundles and induction

Let $E \to B$ be a smooth real G-vector bundle over a smooth G-manifold B. The total space of its real projective bundle, denoted $P(E)$, is a smooth G-manifold consisting of 1-dimensional linear subspaces in the fibers of E. The purpose of this section is to prove

Proposition 2.1. *Let F be a closed smooth G-manifold whose connected components are of dimension $\leqslant 1$ and let $N \to F$ be a smooth G-vector bundle over F. If the fiber dimension of the bundle is less than or equal to one over all 1-dimensional connected components of F, then $P(N \oplus \underline{\mathbb{R}})$ is algebraically realized.*

We recall the notion of strongly algebraic G-vector bundle. Let Ξ be an orthogonal G-module and let $\mathrm{End}(\Xi)$ be a real G-module consisting of endomorphisms of Ξ. For a natural number k we define

$$G_{\mathbb{R}}(\Xi, k) = \{P \in \mathrm{End}(\Xi) \mid P^2 = P, {}^tP = P, \text{trace } P = k\},$$
$$E_{\mathbb{R}}(\Xi, k) = \{(P, v) \in G_{\mathbb{R}}(\Xi, k) \times \Xi \mid Pv = v\}.$$

These are (nonsingular) real algebraic G-varieties. Since $P \in G_{\mathbb{R}}(\Xi, k)$ is an orthogonal projection of Ξ with rank k, the image of P is a k-dimensional linear subspace of Ξ. Hence $G_{\mathbb{R}}(\Xi, k)$ is a Grassmann manifold and $\mathrm{E}_{\mathbb{R}}(\Xi, k) \to \mathrm{G}_{\mathbb{R}}(\Xi, k)$ is a k-dimensional universal G-vector bundle with the obvious projection.

Let X be a real algebraic G-variety and $f: X \to G_{\mathbb{R}}(\Xi, k)$ an equivariant entire rational map. Here entire rational map means that each coordinate function of f is a rational function (we view $G_{\mathbb{R}}(\Xi, k) \subset \mathrm{End}(\Xi) = \mathbb{R}^{(\dim \Xi)^2}$, so the coordinate functions of f make sense). Then the G-vector bundle induced by f from the universal G-vector bundle, denoted by $E_f \to X$, is called *strongly* (*or strictly*) *algebraic* [2,7,11].

Proposition 2.2 [7]. *Let $E_f \to X$ be a strongly algebraic G-vector bundle. Then $P(E_f)$ is a real algebraic G-variety. Moreover, if X is nonsingular, then so is $P(E_f)$.*

Here are two examples of strongly algebraic G-vector bundles.

Lemma 2.3. *Let X be a real algebraic G-variety and W be a real G-module. Then the product bundle $X \times W \to X$ is strongly algebraic.*

Lemma 2.4. *The G-line bundles $S(V) \times \delta \to S(V)$ and $S(V) \times_{\mathbb{Z}_2} \delta \to P(V)$ in Proposition* 1.1 *are strongly algebraic.*

Proof. By Lemma 2.3 it suffices to treat the latter bundle. Consider the map

$$\Psi : S(V) \times_{\mathbb{Z}_2} \delta \to P(V) \times (V \otimes \delta)$$

given by

$$\Psi([x, y]) = ([x], x \otimes y)$$

where $x \in S(V)$, $y \in \delta$ and [ ] denotes the equivalence class. We note that $P(V) = P(V \otimes \delta)$ as δ is 1-dimensional and that $P(V \otimes \delta) = G_{\mathbb{R}}(V \otimes \delta, 1)$. Then one can see that the image of Ψ lies in $E_{\mathbb{R}}(V \otimes \delta, 1)$ and that Ψ is an equivariant bundle map to $E_{\mathbb{R}}(V \otimes \delta, 1)$. The map Ψ induces the identity map on the base spaces, which is obviously equivariant and entire rational. This proves that $S(V) \times_{\mathbb{Z}_2} \delta \to P(V)$ is strongly algebraic. $\square$

We recall a result on induction.

Proposition 2.5 [7]. *Let H be a closed subgroup of G of finite index. If a closed smooth H-manifold P is algebraically realized, then the G-manifold $G \times_H P$ is algebraically realized, where $G \times_H P$ denotes the H-orbit space of $G \times P$ by the H-action defined by $(g, x) \to (gh^{-1}, hx)$ for $h \in H$.*

Proof of Proposition 2.1. If two smooth G-manifolds are algebraically realized, then so is their disjoint union. Hence we may assume that F is the G-orbit of a connected component C in F. Let H be the subgroup of G which preserves C. The restricted bundle $N \mid C \to C$ is an H-vector bundle and $P(N \oplus \underline{\mathbb{R}}) = G \times_H P(N \mid C \oplus \underline{\mathbb{R}})$. Since F is compact, it has only finitely many connected components; so H has a finite index in G. Hence it suffices to prove that the H-manifold $P(N \mid C \oplus \underline{\mathbb{R}})$ is algebraically realized by Proposition 2.5.

Since C is a point or a circle, the H-action on C is linear. Hence we may assume that C is a nonsingular real algebraic H-variety. By Lemma 2.3, $\underline{\mathbb{R}}$ is strongly algebraic. It is not difficult to see that the Whitney sum of strongly algebraic H-vector bundles is again strongly algebraic [7,11]. Thus it suffices to prove that $N \mid C \to C$ is strongly algebraic by Proposition 2.2. But it follows from Lemmas 2.3 and 2.4. □

As remarked before, if two smooth G-manifolds are algebraically realized, then so is their disjoint union. This together with Proposition 2.5 implies that it suffices to prove Theorem 0.1 for *connected* smooth G-manifolds.

3. A special case

From now until the end of Section 5, M will be a connected closed smooth manifold of dimension 2 with an effective smooth G-action. We prove that M is algebraically realized step by step. In this section we assume that G has a central element τ of order 2. The following lemma is well known and holds for a closed smooth G-manifold of any dimension. We shall give a proof for the reader's convenience.

Lemma 3.1. *M is G-cobordant to $P(\nu(M^\tau) \oplus \underline{\mathbb{R}})$.*

Proof. Let M_0 be the complement of a G-invariant open tubular neighborhood of M^τ in M. Since the action of τ on M_0 is free and τ is in the center of G, $M_0 \times_\tau D^1$ is a compact smooth G-manifold where D^1 is the unit disk of $\mathbb{R}$ and τ acts on D^1 as multiplication by -1. We note that

$$\partial\left(M_0 \times_\tau D^1\right) = (\partial M_0) \times_\tau D^1 \cup M_0 \times_\tau \partial D^1.$$

Since $M_0 \times_\tau \partial D^1$ is equivariantly diffeomorphic to M_0, $M_0 \times_\tau D^1$ contains M_0 in the boundary. On the other hand $M \times [0, 1]$ also contains M_0 in the boundary. We paste together $M \times [0, 1]$ and $M_0 \times_\tau D^1$ along M_0. It is not difficult to see that the resulting compact smooth G-manifold gives a G-cobordism between M and $P(\nu(M^\tau)\oplus\underline{\mathbb{R}})$. □

It follows from Proposition 2.1 that $P(\nu(M^\tau)\oplus\underline{\mathbb{R}})$ is algebraically realized. Hence M is algebraically realized by Theorem 0.2. Thus we have proved

Proposition 3.2. *If G has a central element of order 2 (e.g. if G is a 2-group), then any closed smooth G-manifold of dimension 2 is algebraically realized.*

4. The case where G is finite

In this section we prove that M is algebraically realized when G is finite. Let G_x denote the isotropy group at $x\in M$.

Lemma 4.1. *Unless G_x is trivial, $\mathbb{Z}_2$ or $\mathbb{Z}_2\times\mathbb{Z}_2$, the G_x-manifold $P(T_xM\oplus\mathbb{R})$ has only one G_x-fixed point and the isotropy groups at the other points are 2-groups.*

Proof. Since the G_x-action on T_xM is linear and effective, we view G_x as a subgroup of $O(2)$. Let $[u, v]$ be a point of $P(T_xM\oplus\mathbb{R})$, where $u\in T_xM$ and $v\in\mathbb{R}$. Suppose $[u, v]$ is fixed by $g\in G_x$, i.e., $[gu, v]=[u, v]$. Clearly $[0, 1]$ is a G_x-fixed point. In the following we assume $u\neq 0$ and g is not the identity element.

If $v\neq 0$, then $gu=u$ which implies that g is a reflection as $u\neq 0$. Since such a reflection in $O(2)$ is unique, the isotropy group at $[u, v]$ ($u\neq 0$, $v\neq 0$) is trivial or $\mathbb{Z}_2$. If $v=0$, then $gu=u$ or $gu=-u$, which implies that g is a reflection or acts through multiplication by -1. Hence the isotropy group at $[u, 0]$ ($u\neq 0$) is trivial, $\mathbb{Z}_2$ or $\mathbb{Z}_2\times\mathbb{Z}_2$ because G_x is a subgroup of $O(2)$. This proves the lemma. □

Suppose G_x is not a 2-group. Then $P(T_xM\oplus\mathbb{R})$ has only one G_x-fixed point and the isotropy groups of the other points are 2-groups by Lemma 4.1. We take connected sum of M with $G\times_{G_x}P(T_xM\oplus\mathbb{R})$ equivariantly along the G-orbit of x. Let M' be the resulting closed smooth G-manifold. The isotropy group G_x does not appear for M' although we may create isotropy groups which are 2-groups. Moreover M' is G-cobordant to the disjoint union of M and $G\times_{G_x}P(T_xM\oplus\mathbb{R})$ and hence M is G-cobordant to the disjoint union of M' and $G\times_{G_x}P(T_xM\oplus\mathbb{R})$ because 2-copies of a closed smooth G-manifold are equivariantly null-cobordant. Since G is finite and $\dim M=2$, there are only finitely many points in M whose isotropy groups are not 2-groups. We repeat the above procedure finitely many times until we get a closed smooth G-manifold N all of whose isotropy groups are 2-groups.

It follows from Propositions 2.2 and 2.5 that $G\times_{G_x}P(T_xM\oplus\mathbb{R})$ is algebraically realized. Thus it suffices to prove that N is algebraically realized by Theorem 0.2. Since the isotropy groups of N are only 2-groups, a result of Stong [16, Corollary 7.1] says that there is a closed smooth G_2-manifold Q such that $G\times_{G_2}Q$ is

G-cobordant to N, where G_2 denotes a 2-Sylow subgroup of G. It follows from Proposition 3.2 that Q is algebraically realized. Hence $G \times_{G_2} Q$ is algebraically realized by Proposition 2.5 and N is algebraically realized by Theorem 0.2.

5. The case where G is infinite

In this section we treat the case where the dimension of G is positive. A connected closed smooth manifold M of dimension 2 which admits a smooth action of G with positive dimension is S^2, $\mathbb{R}P^2$, T^2 or the Klein bottle. This is seen as follows. Since $\dim G > 0$, a maximal torus of G is nontrivial; so G contains SO(2). As is well known $\chi(M) = \chi(M^{\mathrm{SO}(2)})$ [3, 10.9 in Chapter III] where χ denotes the Euler characteristic and $M^{\mathrm{SO}(2)}$ the SO(2)-fixed point set. Since $\dim M = 2$ and the action is effective, $M^{\mathrm{SO}(2)}$ consists of isolated points if it is nonempty. Hence $\chi(M^{\mathrm{SO}(2)}) = \chi(M)$ is nonnegative, which means that M is S^2, $\mathbb{R}P^2$, T^2 or the Klein bottle.

Let G_0 be the identity component of G. We consider three different cases.

Case 1: *The case where G_0 is non-Abelian.* It is known (cf. [4]) that a non-Abelian connected compact Lie group cannot act nontrivially on T^2 or the Klein bottle. Hence $M = S^2$ or $\mathbb{R}P^2$. Moreover it is known [15, Theorem 2.0] that any smooth G-action on S^2 is linear, so it is algebraically realized. If $M = \mathbb{R}P^2$, then the smooth G-action on $\mathbb{R}P^2$ lifts to a smooth G-action on S^2 [3, 9.4 in Chapter I]. Since the G-action on S^2 is linear as remarked above and effective, G must be isomorphic to a subgroup of $O(3)$, in particular, $\dim G \leqslant 3$. On the other hand, since G_0 is non-Abelian, $\dim G \geqslant 3$ [3, §6 in Chapter 0]. Hence $\dim G = 3$, i.e., $G_0 = \mathrm{SO}(3)$, or SU(2) [3, §6 in Chapter 0]. Now consider the restricted action of G_0 on M. There is a point x of M whose isotropy group is a proper subgroup of G_0. Since any proper subgroup of G_0 has dimension $\leqslant 1$, the dimension of the G_0-orbit of x is greater than or equal to 2. But M is a closed connected manifold of dimension 2, so the G_0-orbit of x must coincide with M. This means that G_0 (and hence G) acts transitively on M, i.e., M is a homogeneous G-space. On the other hand it is known that any homogeneous G-space is algebraically realized (see the proof of [8, Proposition 2.2]). Thus M is algebraically realized.

Case 2: *The case where G_0 is Abelian and* $\dim G_0 \geqslant 2$. Consider the restricted action of G_0 on M. Since G_0 is Abelian and the action is effective, the principal isotropy group of M (i.e., the minimal one among isotropy groups of M) is trivial. Hence G_0 (and G) acts transitively on M because $\dim G_0 \geqslant 2$, $\dim M = 2$ and M is connected. Since M is a homogeneous G-space, it is algebraically realized.

Case 3: *The case where G_0 is Abelian and* $\dim G_0 = 1$. In this case $G_0 = \mathrm{SO}(2)$. Let τ be the order-2 element in G_0. We claim that τ is in the center of G. Let g be any element in G. Since G_0 is a normal subgroup of G and $\tau \in G_0$, $g^{-1}\tau g$ is in G_0 and of order 2. But an order-2 element in $G_0 = \mathrm{SO}(2)$ is unique, so $g^{-1}\tau g$ must be equal to τ. This means that τ is in the center of G. It follows from Proposition 3.2 that M is algebraically realized.

6. The case where M is noncompact

As mentioned in the introduction, a smooth open (i.e., noncompact and without boundary) G-manifold M must be compactified if it is algebraically realized [13,14]. In this section we prove the converse when dim $M = 2$, so that the proof of Theorem 0.1 will be completed.

We may assume M to be connected as before. Suppose M is compactified. Then there is a compact smooth G-manifold W such that $W - \partial W$ is equivariantly diffeomorphic to M. We can take W to be connected as M is connected. Let $\partial_i W$ be a connected component of ∂W and G_i be the subgroup of G which preserves $\partial_i W$. Since $\partial_i W$ is a circle, we may assume that the action of G_i on it is linear. Hence the action of G_i on $\partial_i W$ extends to a linear action of G_i on the 2-dimensional disk. Let D_i be the 2-dimensional disk with the linear G_i-action whose boundary is equivariantly diffeomorphic to $\partial_i W$. We attach the G-manifold $G \times_{G_i} D_i$ to W equivariantly along the boundary. We repeat this procedure until we get a closed smooth G-manifold, denoted $\overline{W}$.

Since we have proved the Interior Manifold Conjecture for closed smooth G-manifolds of dimension 2 in the previous sections, there is a nonsingular real algebraic G-variety X which is equivariantly diffeomorphic to $\overline{W}$. Let Y be the subset of X consisting of the points in X corresponding to centers of the D_i's. It is a G-invariant finite set and hence a real algebraic G-variety. We note that $X - Y$ is equivariantly diffeomorphic to M. On the other hand, the same argument as in the proof of [8, Lemma, p. 50] shows that $X - Y$ is algebraically realized. Thus M is algebraically realized.

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